Text S3: Forward and Backward Inference in Spatial Cognition

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Augmented model

The model in equation 31 (main text) is equivalent to a state-space model without inputs but with an augmented state vector $\tilde{\boldsymbol{x}}_n = [\boldsymbol{x}_n, \boldsymbol{u}_n]^T$ where

$$\begin{aligned} \tilde{\boldsymbol{x}}_n &= \tilde{\boldsymbol{F}}_n \tilde{\boldsymbol{x}}_{n-1} + \tilde{\boldsymbol{z}}_n \\ \boldsymbol{y}_n &= \tilde{\boldsymbol{G}}_n \tilde{\boldsymbol{x}}_n + \boldsymbol{e}_n \end{aligned} \tag{1}$$

with $\tilde{\boldsymbol{z}}_n \sim N(\tilde{\boldsymbol{z}}_n; \boldsymbol{0}, \tilde{\boldsymbol{Q}})$ and $\boldsymbol{e}_n \sim N(\boldsymbol{e}_n; \boldsymbol{0}, \boldsymbol{R})$ and

$$\tilde{\boldsymbol{F}}_n = \begin{bmatrix} \boldsymbol{F}_n & \boldsymbol{H}_n \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \tag{2}$$

$$\tilde{\boldsymbol{G}}_n = \begin{bmatrix} \boldsymbol{G}_n & \boldsymbol{0} \end{bmatrix} \tag{3}$$

and

$$\tilde{\boldsymbol{Q}} = \begin{bmatrix} \boldsymbol{Q}_x & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{Q}_u \end{bmatrix} \tag{4}$$

We assume the initial distributions over state and controls are Gaussian with means $\boldsymbol{m}_0$, $\boldsymbol{r}_0$ and covariances $\boldsymbol{P}_0$ and $\boldsymbol{B}_0$. The mean and covariance of the initial augmented state are therefore $\tilde{\boldsymbol{m}}_0 = [\boldsymbol{m}_0, \boldsymbol{r}_0]^T$ and

$$\tilde{\boldsymbol{P}}_0 = \begin{bmatrix} \boldsymbol{P}_0 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{B}_0 \end{bmatrix} \tag{5}$$

We can therefore implement forward and backward inference as described in the main text but using the augmented quantities $\tilde{\boldsymbol{F}}_n$, $\tilde{\boldsymbol{G}}_n$, $\tilde{\boldsymbol{Q}}$ and $\tilde{\boldsymbol{P}}_0$ instead of $\boldsymbol{F}_n$, $\boldsymbol{G}_n$, $\boldsymbol{Q}$ and $\boldsymbol{P}_0$. Additionally, the $\boldsymbol{H}_n$ term in equation 23 (main text) is set to zero as the input dependence is already incorporated in $\tilde{\boldsymbol{F}}_n$. The desired control density is a Gaussian

$$p(\boldsymbol{u}_n | \boldsymbol{x}_1, \boldsymbol{Y}_N) = \mathsf{N}(\boldsymbol{u}_n; \hat{\boldsymbol{u}}_n, \hat{\boldsymbol{B}}_n) \tag{6}$$

and the quantities $\hat{\boldsymbol{u}}_n$ and $\hat{\boldsymbol{B}}_n$ from backward inference, can then be read off from the corresponding posterior estimates of the augmented states. In the absence of correlations between inputs and hidden states the update formulae for $\hat{\boldsymbol{u}}_n$ and $\hat{\boldsymbol{B}}_n$ have the simpler form shown in equation 34 in the main text.